Kinetic Theory 1 / Probabilities

1. Motivations: statistical mechanics and fluctuations
2. Probabilities
3. Central limit theorem
Main concept introduced in first half of this chapter

A) Temperature
B) Pressure
C) Entropy
D) Diffusion
E) Probabilistic description
In class

Reading check

Most important: Probability description
The need for statistical mechanics
How to describe large systems

In this course we will try to make the connection between the macroscopic description of thermodynamics and the microphysics behind it. The prototype that we will use is a volume of gas.

http://cosmology.berkeley.edu/Classes/S2009/Phys112/Diffusion/GaussDiffusion.html

Our system are large!

Avogadro number $N = 6.02 \times 10^{23}$ / mole (=22.4 liter of gas, =12 gram of Carbon)

Very large number of particles $10^{28}$ in $500 \text{m}^3$

At finite temperature not at rest

e.g Velocity $\approx 300 \text{m/s}$ for air molecules at 300K

\[
\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}k_BT \quad \text{with} \quad T \approx 300K \\
k_B = 1.38 \times 10^{-23} \text{ J/K} \\
m \approx 32 \times 10^{-3} / 6.0210^{23} \text{ kg}
\]

Constant collisions

Velocity magnitude and direction change at each
e.g. gas in room: Mean free path $\lambda \approx 0.1 \mu\text{m}$

\[
\lambda = \frac{1}{\sigma n} \approx \frac{1}{\pi d^2 n} \quad \text{with} \quad d = 3 \times 10^{-10} \text{ m} \\
n = 6.02 \times 10^{23} / 0.0224 \text{ m}^{-3}
\]
Ideal Gas

What is an ideal gas?

A) An ordinary gas such as air
B) Something that obeys $PV = NkT$
C) Particle probability distributions are independent
D) Independent particles without interactions
E) Particle systems where interaction time $\ll$ mean free time between collisions
In class

Ideal gas

Answers C or E
⇒ **Probabilistic description. 1 particle**

**Classically:** We have to track both position and velocities (or momentum). We will introduce

\[ f(\vec{x}, \vec{p})d^3xd^3p \]

probability distribution on phase space = position × momentum

**Quantum:** probability \( p_i \) of single particle to be in state \( i \)

**\( N \) particles:**

In many cases, a good approximation is to treat them as independent

**We will call this an Ideal Gas**


In terms of formalism, this corresponds to taking the product of probability distribution:

\[ \prod_{\text{Particles } i} f(\vec{x}_i, \vec{p}_i)d^3x_i d^3p_i \]

In general, there are correlations between particles ⇒ not independent. Example: Van der Waals gas, liquids
Temperature

What is temperature?

A: A measurement of how hot a system is (as measured by a thermometer)

B: A measurement of the random kinetic energy of particles in the system

C: A measurement of the mean energy of excitations in the system

D: \[ \langle \frac{1}{2}mv^2 \rangle = \frac{3}{2} k_b T \]

E: A measurement of the “kicks” to the system
Closest answer C

2 reasons

1) Usually more degrees of freedom than spatial degrees of freedom

   Spin (e.g., ferromagnetism)
   Diatomic molecules: rotation and vibration

\[ \langle \varepsilon \rangle = k_B T = \tau \]

\[ \varepsilon \approx k_B T \]

2) Other effects: Quantum gas: Fermi Dirac exclusion principle means that we have to stack

\[ f(\varepsilon, \tau) \]

\[ \varepsilon_F \]

\[ \varepsilon \]
Thermodynamics: Macroscopic Quantities

Temperature

A measure of the mean energy of excitations of the system
e.g., in a classical gas, excitatioon = random velocity of particles

\[ \left\langle \frac{1}{2} m v^2 \right\rangle = \frac{3}{2} k_B T \]

We do not care about coherent motion, e.g., gas bulk velocity.
Many more degrees of freedom
- Spins, rotation, vibration, broken Cooper pairs

We are only interested what changes when we heat up the system.
=> We will have to understand how to calculate mean energy

\[ \langle \mathcal{E} \rangle \equiv \frac{\int \mathcal{E}(p) f(\vec{x}, \vec{p}) d^3 x d^3 p}{V} \]
Pressure

What is pressure?

What is the most accurate answer?

A: A measure of the “kicks” that particles give each other during collisions

B: \( \frac{NkT}{V} \)

C: The average force per unit area on the walls of the container

D: A measure of the energy density

E: The force per unit area on the walls of the container
C/D are the closest answers

B: is a formula of limited applicability
A <-> D are related to the microphysics of collisions
C is better than E because of the fluctuations
We will show that quite generally

non relativistic \[ u = \frac{3}{2} \frac{N}{V} k_b T \equiv \frac{3}{2} \frac{N}{V} \tau \Rightarrow P = \frac{N}{V} \tau \]

= same pressure as thermodynamic definition \[ \frac{T \partial S}{\partial V} \bigg|_{U,N} \]

ultra relativistic \[ pv = \varepsilon \Rightarrow P = \frac{1}{3} u \]
Macroscopic quantities 2

Pressure

2 possible definitions
1) The simplest
“The average force per unit area on the walls of the container”

2) A measure of the kicks that particles give each other per unit time= related to the mean energy density

\[
\text{non relativistic} \quad u = \frac{3}{2} \frac{N}{V} k_b T = \frac{3}{2} \frac{N}{V} \tau \Rightarrow P = \frac{N}{V} \tau
\]

\[
\text{ultra relativistic} \Rightarrow pv = \varepsilon \Rightarrow P = \frac{1}{3} u
\]
In class

Entropy

What is entropy of a system?

A: A measure of the state of disorder of a system

B: A measure of the quantity of heat in the system

C: A measure of the disordered energy in the system
A

With modern definition

\[ \sigma = - \sum_i p_i \log p_i \]

= 0 if in one state
= max if all states are equiprobable

B C No

\[ \delta Q = TdS \]
**Macroscopic quantities**

**Thermodynamics identity**

At equilibrium

\[ dU = -TdS - pdV + \mu dN \equiv -\tau d\sigma - pdV + \mu dN \]

with \( U \) = energy of the system, \( V \) = volume, \( N \) = number of particles

\( \tau \equiv k_bT \) = temperature \( S \equiv \frac{\sigma}{k_b} \) = entropy and \( \mu \) = chemical potential

**Entropy**

A measure of the disorder. We will define it as

\[ \sigma = -\sum_i p_i \log p_i \]

Note that this requires discrete states

Measures how unequal the probabilities are

Maximum when all probabilities are equal

\[ \sigma = -\sum_i p_i \log p_i = \frac{1}{g} = \log g \]

\( g \) = number of states in the system
How does a system reaches equilibrium

Diffusion

Consider the gas in this room
Introduce a puff of CO gas in the center of the room
diffuses out \(\leq\) collisions

starting from blue curve evolves towards uniform distribution
http://cosmology.berkeley.edu/Classes/S2009/Phys112/Diffusion/GaussDiffusion.html

True also in probability space

An isolated system tends to evolve towards maximum flatness

All states equally probable! Maximum entropy!
Fluctuations

Because of finite number of particles, fluctuations

Example: Pressure of an ideal gas

http://cosmology.berkeley.edu/Classes/S2009/Phys112/idealGas/idealGas.html

Note that when the number of particles increase the fluctuations become relatively smaller

\[ \frac{\Delta P}{P} \approx \frac{1}{\sqrt{N}} \leftarrow \text{Central limit theorem} \]

⇒ Thermodynamics is a good approximation
Probabilities

See e.g.,
http://en.wikipedia.org/wiki/Probability
http://mathworld.wolfram.com/Probability.html
In class

Question 1: Roulette

In the US: 38 slots. What is the probability that the ball lands in one particular slot?

A Don’t know
B 1/35
C 1/38
Question 1: Roulette

In the US: 38 slots. What is the probability that the ball lands in one particular slot?

A Don’t know
B 1/35 You get only 35 times your bet
C 1/38
**Probability, Moments**

**Probabilistic Event:** occurs in a "uncontrolled" way described by **Random Variable**
≠ parameter = fixed (even if unknown)

1) **Discrete Case:** consider “bins”

- n boxes labelled s (e.g. roulette)
- "Drawing" of N objects
- N_s "land" in box s

**Probability** = measure on sets defined by the limit of the ratio of number of outcome s to the total number of draws

\[
P(s) = \lim_{N \to \infty} \frac{N_s}{N} \geq 0 \quad \text{Normalized} \equiv \sum_s P(s) = 1
\]

An important example: **Histograms:** make boxes of width \( \Delta x \)

As the total number of events

\[
N \to \infty \quad \frac{N_i}{N} \to P_i
\]

See e.g.: [http://cosmology.berkeley.edu/Classes/S2009/Phys112/Diffusion/GaussDiffusion.html](http://cosmology.berkeley.edu/Classes/S2009/Phys112/Diffusion/GaussDiffusion.html)
Probability

Overlapping possible outcome

\[ P(A = \{s_i\} \cup B = \{s_j\}) = \sum P(s_i) + \sum P(s_j) = P(A) + P(B) - P(A \cap B) \]

\[ P(A \cap B) = \sum_{i = j}^{\text{Common}} P(s_i) = 0 \text{ if no common region of outcomes} \]

Combined with positivity and normalization => Measure on set
Can we determine probabilities experimentally?

A: Yes
B: No
In class

**Probabilities**

**Answer is B: NO**

\[ P(s) = \lim_{N \to \infty} \frac{N_s}{N} \geq 0 \quad \text{Normalized} \equiv \sum_s P(s) = 1 \]

But \( \frac{N_s}{N} \) is an estimator of \( p_s \)

\[ \frac{N_s}{N} \to p_s \]

The error is of the order of \( \sqrt{\frac{p_s(1-p_s)}{N}} \)

**We can predict probabilities theoretically**

e.g., by symmetry arguments

But system may be imperfect!
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Conditions on probabilities

Consider 5 possible outcomes

Which sequence of numbers can be a set of probabilities for these outcomes?

A: 0.1 0.5 0.3 0.2 0.1
B: 0.1 0.7 0.05 0.1 0.05
C: 0.3 -0.1 0.06 0.15 0.005
In class

**Conditions on probabilities**

B is correct

Probabilities have to be positive and normalized

\[ P(s) = \lim_{{N \to \infty}} \frac{N_s}{N} \geq 0 \quad \text{Normalized} \equiv \sum_s P(s) = 1 \]
Probability, Moments 2

Moments

Consider a real function $y(s)$ defined on the random sets $s$

Mean = Expectation Value (≠ Experimental Average)

$$< y > = \sum_s y(s) \cdot P(s)$$

Variance

$$\sigma_y^2 = < (y - < y > )^2 > = < y^2 > - < y >^2$$

root mean square (r.m.s.)

= standard deviation $\sigma_y = \sqrt{\sigma_y^2}$

Measurement of the width of the distribution

Independence

Consider several random variables (in general they have some causal connection)

$$x, y \quad P(x \perp y) = P(x) \cdot P(y) \quad \text{and} \quad P(y \perp x) = P(y) \cdot P(x)$$

Independence

$$P(x,y) = P(x)P(y) \Leftrightarrow P(x/y) = P(x) \Leftrightarrow P(y/x) = P(y)$$

Correlation

$$\rho_{xy} = \frac{< (x - < x > ) \cdot (y - < y > ) >}{\sigma_x \sigma_y}$$

= 0 if independent (but in general not the reverse!)
In class

Independence

Which of these distributions appear to be independent?

Which of these distributions seem to have zero correlation coefficient?
In class

Independence

Which of these distributions appear to be independent? **C**

Which of these distributions seem to have zero correlation coefficient? **C and D**

We say “appear” or “seem” because we can have only an estimate
Discrete Probabilities

2 important examples

• Poisson Distribution
  Applies to the case where we count the number of independent random events in a drawing or an experiment (e.g. number of events in a given time interval, when the events are independent and occurring at constant rate, for example radioactive decays),
  The probability of obtaining exactly \( n \) events when we expect \( m \) is
  \[
  \text{Prob}(n) = e^{-m} \frac{m^n}{n!}
  \]
  \( \mu = m \)
  \( \sigma^2 = m \)  Statistical fluctuations \( \propto \sqrt{m} \)

• Binomial distribution
  Applies to the case when there are two possible outcomes 1 and 2 of probabilities \( p \) and \( 1-p \) (e.g. number of events in an histogram bin: either the event falls in the bin or outside, or in case of spin 1/2 particles, number of spins up).
  The probability of obtaining exactly \( n \) outcomes 1 in \( N \) trials (i.e. \( N \) total number of events)
  \[
  \text{Prob}(n/N) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}
  \]
  \( \mu = Np \)
  \( \sigma^2 = Np(1-p) \)  Statistical fluctuations \( \propto \sqrt{N} \)
2) Continuous Case

Infinitesimal box $dx$

$P(dx) = f(x)dx$

$\int f(x)dx = 1$

$< y >= \int y(x)f(x)dx$

$< y^2 >= \int y^2(x)f(x)dx$

$\sigma_y^2 = \int (y(x) - < y >)^2 f(x)dx$

Examples:

1) Path length of a particle on mean free path $\lambda$

Prob interaction between $l$ and $l + dl$:

$f(l)dl = \frac{1}{\lambda} \exp\left(-\frac{l}{\lambda}\right)dl$

2) Gaussian (= Normal)

$f(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

$\mu$ and $\sigma^2$ are indeed the mean and variance

3) Maxwell: ideal gas => The probability for a given particle to have velocity $v$ in solid angle cone $d\Omega$ is

$f(v)dvd\Omega = \left(\frac{M}{2\pi\tau}\right)^\frac{3}{2} \exp\left(-\frac{Mv^2}{2\tau}\right) v^2 dv d\Omega$ with $d\Omega = d\cos\theta d\phi$

This assumes implicitly polar coordinates

\[ v_x = v \sin\theta \cos\phi \]
\[ v_y = v \sin\theta \sin\phi \]
\[ v_z = v \cos\theta \]
**Summary of Moments**

**Moments**

\[
\langle y(x) \rangle \equiv \int_{-\infty}^{\infty} y(x)f(x) \, dx
\]

In particular \( \mu_1 = \langle x \rangle \equiv \int_{-\infty}^{\infty} x f(x) \, dx \equiv \mu \)

\[
\mu_2 = \langle x^2 \rangle \equiv \int_{-\infty}^{\infty} x^2 f(x) \, dx \text{ etc...}
\]

**Centered moments**

\[
\mu'_1 = \langle x \rangle
\]

\[
\mu'_2 = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2
\]

\[
\mu'_3 = \langle (x - \langle x \rangle)^3 \rangle \text{ etc...}
\]

**Cumulants**

\[
\kappa_i = i^{\text{th}} \text{ cumulant}
\]

\[
\kappa_1 = \langle x \rangle
\]

\[
\kappa_2 = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2
\]

\[
\kappa_3 = \langle (x - \langle x \rangle)^3 \rangle \quad \frac{\kappa_3}{\sigma^{3/2}} = \text{skewness}
\]

\[
\kappa_4 = \langle (x - \langle x \rangle)^4 \rangle - 3\sigma^4 \quad \frac{\kappa_4}{\sigma^4} = \text{kurtosis}
\]
What is the mean $\mu$?

$$f(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx$$
What is the mean $\mu$?

\[ f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2} \, dx \]
In class

Normal distribution

What is the standard deviation $\sigma$?

$$f(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx$$
Normal distribution

What is the standard deviation $\sigma$?

$$f(x)dx = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} dx$$

$A = \text{Full width at half maximum} \equiv FWHM \approx 2.3\sigma$
In class

**Maxwell distribution**

\[
f(v)dv \int d\Omega = \left( \frac{M}{2\pi \tau} \right)^2 \exp\left( - \frac{Mv^2}{2\tau} \right) v^2 dv \ 4\pi
\]

\[M = 1, \tau = 3\]

![Graph showing the Maxwell distribution with points A, B, and C.]

**What is the mean \( \mu \)?**
In class

Maxwell distribution

\[ f(v)dv \int d\Omega = \left( \frac{M}{2\pi\tau} \right)^{\frac{3}{2}} \exp\left( -\frac{Mv^2}{2\tau} \right)v^2dv \ 4\pi \]

\[ M = 1, \tau = 3 \]

What is the mean \( \mu \)?
In class

Maxwell distribution

\[ f(v)dv \int d\Omega = \left( \frac{M}{2\pi \tau} \right)^{\frac{3}{2}} \exp\left( -\frac{Mv^2}{2\tau} \right) v^2 dv \ 4\pi \]

\[ M = 1, \tau = 3 \]

What is the standard deviation \( \sigma \)?
Maxwell distribution

\[ f(v)dv \int d\Omega = \left( \frac{M}{2\pi \tau} \right)^{\frac{3}{2}} \exp\left( -\frac{Mv^2}{2\tau} \right) v^2 dv \quad 4\pi \]

\[ M = 1, \tau = 3 \]

What is the standard deviation \( \sigma \)?
Independence/Correlation

Two variables are said to be independent if and only if:

\[ f(x, y)dx\,dy = g(x)h(y)dx\,dy \]

Example: Maxwell distribution

\[ g(v_x, v_y, v_z)dv_x\,dv_y\,dv_z = \left( \frac{M}{2\pi\tau} \right)^{\frac{3}{2}} \exp \left[ -\frac{M(v_x^2 + v_y^2 + v_z^2)}{2\tau} \right] dv_x\,dv_y\,dv_z \Rightarrow v_x, v_y, v_z \text{ are independent} \]

Correlation:

Example: Normal distribution

\[ f(x, y)dx\,dy = \frac{\sqrt{1-\rho^2}}{2\pi\sigma^2} \exp \left( -\frac{x^2 - 2\rho xy + y^2}{2\sigma^2} \right)dx\,dy \]
Successive approximation in Statistics

Successive moments

• “macroscopic quantities” are usually means
• Root mean square dispersion: Fluctuations around the mean
• Note that

\[ \mu_n(\lambda x) = \left\langle (\lambda x - \lambda \langle x \rangle)^n \right\rangle = \lambda^n \mu_n(x) \]

Characteristic function and cumulants (optional)

\[ f(x)\,dx \leftrightarrow \text{Fourier transform} \quad \phi(t) = \int \exp(-itx)f(x)\,dx \]

Expanding

\[ \exp(-itx) = 1 - itx + \frac{(-it)^2}{2!}x^2 + \frac{(-it)^3}{3!}x^3 + .. \]

\[ \phi(t) = 1 - it\langle x \rangle + \frac{(-it)^2}{2!}\langle x^2 \rangle + \frac{(-it)^3}{3!}\langle x^3 \rangle + .. \]

If we take the log

\[ \log(\phi(t)) = 1 - it\kappa_1 + \frac{(-it)^2}{2!}\kappa_2 + \frac{(-it)^3}{3!}\kappa_3 + .. \]

\( \kappa_i = i^{th} \) cumulant

\( \kappa_1 = \langle x \rangle \)

\( \kappa_2 = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2 \)

\( \kappa_3 = \left\langle (x - \langle x \rangle)^3 \right\rangle \quad \frac{\kappa_3}{\sigma^{3/2}} = \text{skewness} \)

\( \kappa_4 = \left\langle (x - \langle x \rangle)^4 \right\rangle - 3\sigma^4 \quad \frac{\kappa_4}{\sigma^4} = \text{kurtosis} \)

for Gaussian \( \kappa_n = 0 \) for \( n \geq 3 \)
Sum and Scaling

**Sum of random variables**

\[< y + z > = < y > + < z > \]

\[\sigma_{y+z}^2 = \sigma_y^2 + \sigma_z^2 + 2\rho\sigma_y\sigma_z\]

**if independent**

\[\rho = 0 \Rightarrow \sigma_{y+z}^2 = \sigma_y^2 + \sigma_z^2\]

More generally, if \(x = y + z\), with \(y, z\) independent

\(y\) distributed as \(f(y)dy\), \(z\) distributed as \(g(z)dz\)

the distribution of \(x\) is \(h(x)dx = dx\int f(y)g(x-y)dy = f*g(x)dx\)

\(\Rightarrow\) the cumulants add!

**Optional**

Proof (optional)

Convolution in original space is equivalent to product in Fourier space

Hence for a sum the characteristic functions multiply

and the logs of characteristic functions add!

**Multiplying by a constant**

\[\left<(\lambda y)^n\right> = \lambda^n \left<y^n\right> \Rightarrow \sigma_{\lambda y}^2 = \lambda^2\sigma_y^2\]

\[\kappa_n(\lambda y) = \lambda^n\kappa_n(y)\]
Central Limit Theorem

Consider N independent random variables

\[ x_i < x_i = \mu_i \quad \sigma^2_{x_i} = \sigma^2_i \]

Consider average

\[ A = \frac{1}{N} \sum_{i=1}^{N} x_i \]

Theorem: Asymptotically, random variable A has a Gaussian distribution

\[ f_N(A) dA \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left(\frac{A-\mu}{\sigma}\right)^2} dA \]

\[ \mu = \frac{1}{N} \sum_{i=1}^{N} \mu_i \quad \sigma^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sigma^2_i \]

Proof (optional): Look at cumulants which add!

Consequences

- Initial distribution not important

- Experimental errors are Gaussian

provided that sum of large number of small errors:

\[ m = g(m_1, m_2, m_3, \ldots) = g(true) + \sum_i \frac{\partial g}{\partial m_i} \delta m_i \]

- Distribution are getting very narrow

If \( \sigma_i \) are bounded,

\[ \sigma^2 \propto \frac{1}{N} \]

Simulation: [http://cosmology.berkeley.edu/Classes/S2009/Phys112/CentralLimit/MultipleCoinToss.html](http://cosmology.berkeley.edu/Classes/S2009/Phys112/CentralLimit/MultipleCoinToss.html)
Apply also to sums

Consider

\[ B = \sum_{i=1}^{N} x_i \] with \( \{x_i\} \) independent of mean \( \mu_0 \) and standard deviation \( \sigma_0 \)

What is the asymptotic distribution of \( B \)?

A: \[ f_N(B)dB \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(B-\mu)^2}{\sigma^2}} dB \] with \( \mu = \frac{\mu_0}{N} \) \( \sigma^2 = \frac{\sigma_0^2}{N^2} \)

B: \[ f_N(B)dB \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(B-\mu)^2}{\sigma^2}} dB \] with \( \mu = \mu_0 \) \( \sigma^2 = \frac{\sigma_0^2}{N} \)

C: \[ f_N(B)dB \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(B-\mu)^2}{\sigma^2}} dB \] with \( \mu = N\mu_0 \) \( \sigma^2 = N\sigma_0^2 \)

D: No definite answer (\( B \to \pm \text{infinity} \))
In class

Apply also to sums

Consider

\[ B = \sum_{i=1}^{N} x_i \] with \( \{x_i\} \) independent of mean \( \mu_0 \) and standard deviation \( \sigma_0 \)

What is the asymptotic distribution of \( B \)?

\[ f_N(B)dB \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2} \frac{(B-\mu)^2}{\sigma^2}} dB \] with \( \mu = N\mu_0 \) \( \sigma^2 = N\sigma_0^2 \)

\( B \) goes to \( \pm \) infinity but its distribution relative width decreases with \( N \)

\[ \iff A = \frac{1}{N} \sum_{i=1}^{N} x_i \quad B = NA \]

\[ f_N(A)dA \xrightarrow{N \to \infty} \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{1}{2} \frac{(A-\mu_1)^2}{\sigma_1^2}} dA \] with \( \mu_1 = \mu_0 \) \( \sigma_1^2 = \frac{\sigma_0^2}{N} \)