Physics 112  
Spring 2006  

Final  

(180 minutes = 180 points)  

Do not be afraid of the number of pages: I attempted both to guide you along the way and to give you enough space. Try to time yourself. There is additional credit for the last question of the last problem. You may use four one-sided pages of notes.  

1) Thermodynamic identity (20 points)  

We consider a system with only one species of particles and define with our usual notation the "Grand Potential"  

$$\Omega = F - \mu N = U - \tau \sigma - \mu N$$  

a) (5 points) Write down the thermodynamic identity for $\Omega$.  

Starting e.g. from  

$$dU = -T d\sigma - p dV + \mu dN$$  

$$dF = -\sigma d\tau - p dV + \mu d\mu$$  

$$d\Omega = -\sigma d\tau - p dV + N d\mu$$  

The natural variables are $\tau, V, \mu$  

b) (5 points) Show that when expressed as a function of temperature, volume and chemical potential, the pressure is independent of the volume  

$$p(\tau, V, \mu) = p(\tau, \mu)$$  

(Hint: divide the system into two unequal volumes in equilibrium)  

At equilibrium, $p$ is constant across the sys.  

In particular for two points $$(V_1 \neq V_2)$$  

$$p(\tau, V_1, \mu) = p(\tau, V_2, \mu) \Rightarrow p(\tau, \mu)$$ is independent of the volume  

c) (5 points) Deduce from b) and the fact that $\Omega$ is zero at zero volume that  

$$\Omega = -p V$$  

(Since we have three variables, be careful in defining the path of integration)  

$$\Omega = -\int_0^V p dV + c(\tau, \mu)$$  

$$= -p V + c(\tau, \mu)$$  

But $c(\tau, \mu) = 0$ because $\Omega = 0$ for $V = 0$  

$$\Rightarrow \Omega = -p V$$
d) (2 points) Deduce from this result that the Gibbs Free Energy is equal to

\[ G = F + PV = \mu N \]

\[ F = \mathcal{U} + \mu N \]

\[ G = \mathcal{U} + \mu N + PV \]

\[ \mathcal{U} = -PV \]

\[ \Rightarrow G = \mu N \]

e) (3 points) Check this general result with the expression of \( U, \sigma \) and \( \mu \) for a monoatomic ideal gas. You may remember that

\[ \sigma = N \left[ \log \left( \frac{n_0}{n} \right) + \frac{5}{2} \right] \]

\[ U = \frac{3}{2} N \mathcal{Z} \]

\[ \Rightarrow F = U - \mathcal{Z} \sigma \]

\[ = N \mathcal{Z} \left( \log \frac{n}{n_0} \right) + \left( \frac{3}{2} - \frac{5}{2} \right) N \sigma \]

\[ F = N \mathcal{Z} \left( \log \frac{n}{n_0} - 1 \right) \]

\[ \mu = \mathcal{Z} \log \frac{n}{n_0} \]

\[ \mathcal{U} = F - \mu N \]

\[ = -N \mathcal{Z} = -PV \]

\[ G = F + PV = F + N \mathcal{Z} = N \mathcal{Z} \log \frac{n}{n_0} + \mu N \]
2) Electricity generation and heat pumps (20 points)

Most thermal electric plants (fossil or nuclear fuels) use water vapor as the fluid, which is typically heated to 350°C under pressure. The steam can be condensed typically at a temperature of 35°C.

a) (5 points) What is the maximum efficiency $\eta$ of the plant? ($1^\circ\text{C}=273\text{K}$)

\[
\eta_c = 1 - \frac{T_e}{T_e} = 1 - \frac{273 + 35}{273 + 350} = 0.6\%
\]

b) (3 points) Typical efficiencies are however somewhat lower (35%). How much energy has to be released into the environment to operate a plant delivering an electric power of 1000MW?

\[
\eta = 0.35
\]
\[
\frac{\Delta Q_e}{\Delta t} = \frac{1.86 \times 10^9 \text{Watts}}{\text{s}}
\]

\[
(= \text{J/s})
\]

c) (3 points) A first solution is to use water for a stream, a river or a lake. Environmental regulation limit the increase of temperature of the released water to 3°C. What minimum amount of water (in m³/s) is needed to operate a 1000MW (electric power) plant? (The heat capacity of water is 4.186 $10^3$ J/kg/K)

\[
\text{Volume of water per unit time } = \frac{\Delta V}{\Delta t} \text{ has to be such that }
\]
\[
C \frac{\Delta V}{\Delta t} \Delta T = \frac{\Delta Q_e}{\Delta t}
\]
\[
\frac{\Delta V}{\Delta t} = \frac{\Delta Q_e}{\Delta t} \frac{1}{C \Delta T} = \frac{1.86 \times 10^9}{4.186 \times 10^3 \times 3} = 148 \text{ m}^3/\text{s}
\]
d) (4 points) A better solution is to use cooling towers where the main effect is the evaporation of the water. Typically you can evaporate 5% of the water that you circulate. Knowing that the latent heat of water vaporization is $2.26 \times 10^3$ J/kg, how much water do you need to replenish per second and how much do you need to circulate to operate a 1000 MW plant? Assuming that this water re-condenses on an area of 100 km$^2$, how much rain per year (in inches) does this correspond to? The effect on the local climate is not negligible!

$$\text{Now 1 m}^3 \text{ of evaporated water corresponds to } 2.26 \times 10^3 \text{ J}$$

$$\Rightarrow \frac{\Delta V_{\text{evap}}}{\Delta t} = \Delta q_e \frac{1}{C_L} = \frac{1.6 \times 10^3}{2.26 \times 10^9} \approx 0.82 \text{ m}^3 / \text{s}$$

Total circulation $\frac{\Delta V_c}{\Delta t} = 0.82 \times 86400 \times 365 = 10^8 \times 2.5 \times 10^{-2} = 10^6 / \text{yr}$ is ~ 1.5 M!

c) (5 points) Using electricity produced in thermal for heating purpose is wasteful. In order to get back the efficiency, it has been suggested to use heat pumps, cooling the outside air to heat up the inside of the house. Re-derive rapidly the efficiency of a heat pump and show that indeed for any reasonable temperature range (house temperature $< 25^\circ C$ and outside air $> -50^\circ C$) the (reversible) efficiency is large enough to compensate for the loss during the thermal generation process.

\[
\begin{align*}
\Delta W &= \Delta q_e \left(1 - \frac{\text{T}_h}{\text{T}_e}\right) \\
\Rightarrow \eta &= \frac{\Delta q_e}{\Delta W} = \frac{\text{T}_h}{\text{T}_e - \text{T}_h} \\
\text{put} \quad \text{T}_h &= 25^\circ C + 273 = 298 K \\
\text{T}_e &= 273 - 50 = 223 \\
\eta &= 3.97 > \frac{1}{\eta} = 2.86 \\
\text{Note that if T}_h \text{ decreases or T}_e \text{ increases } \eta \text{ increases.}
\end{align*}
\]
3) Thermal shielding in a cryogenic system (20 points)
Large dilution refrigerators such as the one used by the Cryogenic Dark Matter Search provide $T_0 = 10 \text{mK}$ in a volume of outside surface area of the order of $S = 1 \text{ m}^2$. However they have only a few microwatts of cooling capability.

![Diagram of thermal shielding in a cryogenic system]

a) (10 points) We assume for simplicity that the outer surface is black. Using a detailed balance argument, show that provided the inner volume has a convex outside surface (it does not radiate onto itself), the net radiation power transfer from an outside surface of any shape at temperature $T_1$ is

$$P = \sigma_b \alpha (T_1^4 - T_0^4) S = \sigma_b \alpha T_1^4 S \text{ if } T_1^4 >> T_0^4$$

where $S$ is the surface area of the inner volume and $1 - \alpha$ is its reflectivity. (Hint: Determine the fraction $f$ of the radiation emitted by the outer surface and absorbed by the inner surface through the requirement that the power transfer is equal to zero when the two temperatures are equal)

$$P = \frac{\text{Power emitted by outer surface}}{T_1} - \frac{\text{Power absorbed by inner surface}}{T_1}$$

Net Power

$$P = \text{Power emitted by outer surface} - \text{Power absorbed by inner surface}$$

$$P = \sigma_b \alpha (T_1^4 - T_0^4) S = \sigma_b \alpha T_1^4 S \text{ if } T_1^4 >> T_0^4$$

$$\Rightarrow f = \frac{\text{Power emitted by outer surface}}{\text{Power absorbed by inner surface}}$$

$$\Rightarrow f = \frac{S}{S_{outer}}$$
b) (5 points) Knowing that
\[ \sigma = \frac{c}{4} a_n = 5.67 \times 10^{-8} \text{ W/m}^2/\text{K}^4 \]
and considering a reflectivity of 99%, show that we cannot afford to surround the inner volume with a surface at room temperature (\(T_i = 293\text{K}\)).
\[ \alpha = 1\% \]
\[ P = \alpha \sigma B T_i^4 S = 0.01 \times 5.67 \times 10^{-8} \times (293)^4 \times 1 \]
\[ = 4.2 \text{ W} \]

c) (5 points) While it is clearly acceptable with good reflectivity to have a surface at liquid helium (4K), show that it is necessary to have an additional shield (e.g. at liquid nitrogen 77K) to limit the consumption of liquid helium to a reasonable level (less than 10 liters per day). Compute in particular the consumption in liters of helium per day without and with a liquid N2 shield. We assume a surface area of the system at 4K of \(S_i = 5 \text{ m}^2\) and the same reflectivity of 99%. The latent heat of liquid He is 2.63 \times 10^3 \text{ Joules/liter}.
\[ \frac{1}{T_1} = 4 \text{ K} \]
\[ P = \alpha \sigma B T_i^4 S = 0.01 \times 5.67 \times 10^{-8} \times (293)^4 \times 1 \]
\[ = 1.47 \times 10^{-2} \text{ W} \]
\[ \text{This is ok.} \]

However, \(P_{\text{room temp}} = 21 \text{ W} \)

Consumption of helium \(= \frac{21 \times 86400}{2.63 \times 10^3} = 630 \text{ liters/day} \)

If instead of looking at room temperature to 4K,
level is looked at 77K
\[ P = 0.1 \text{ W} \]

Consumption = 3.38 liters/day
4) Entropy and Free Energy (25 points)

a) (4 points) Let us first consider a system which is isolated and in equilibrium. What is the probability $p_i^0$ of the state $i$ and the entropy $\sigma$ of the system.

From our postulate (the H theorem), the probabilities of all the states are equal, $p_i^0 = \frac{1}{g_t}$, where $g_t$ is the total number of states of the system.

We have $\sum p_i = 1$

b) (5 points) We disturb the system so that it is distributed over its accessible states in accordance to an arbitrary probability distribution $p_i$ which stays normalized:

$$\sum_i p_i = 1$$

We would like to compare its entropy $\sigma$ with its equilibrium entropy $\sigma_0$. Show that

$$\sigma - \sigma_0 = \sum_i p_i \log \left( \frac{p_i^0}{p_i} \right)$$

$$\sigma = -\sum_i p_i \log p_i$$

$$\sigma_0 = -\sum_i p_i^0 \log p_i^0$$

Since $\sum p_i = \sum p_i^0 = 1$, log $p_i$ independent of $i$.

$$\sigma - \sigma_0 = \sum_i p_i \log \left( \frac{p_i^0}{p_i} \right)$$

c) (4 points) Using the fact that $\log \left( \frac{p_i^0}{p_i} \right) < \left( \frac{p_i^0}{p_i} \right) - 1$ show that $\sigma \leq \sigma_0$, with the equality being only obtained if $p_i = p_i^0$ for all $i$.

$$\sigma - \sigma_0 \leq \sum_i p_i \left( \frac{p_i^0}{p_i} - 1 \right) = \sum_i p_i^0 - \sum_i p_i = 0$$

Equality when all the $p_i = p_i^0$.

d) (4 points) We now consider a system that is at equilibrium with a reservoir at the temperature $T$. What is now the probability $p_i^0$ of state $i$ of energy $\varepsilon$?

$$p_i^0 = \frac{1}{\mathcal{Z}} e^{-\frac{\varepsilon_i}{T}}$$

$$\mathcal{Z} = \sum \varepsilon_i e^{-\frac{\varepsilon_i}{T}}$$
c) (5 points) We disturb the system so that it is distributed over its accessible states in accordance to an arbitrary probability distribution \( p_i \) that stays normalized. We can compute the entropy (in terms of the probability distribution) and the mean energy in this disturbed state and therefore the Landau free energy \( F \) of the system (The Landau free energy is the Helmholtz free energy with an arbitrary probability distribution for the states. The temperature is taken as the temperature of the reservoir). Comparing it with the equilibrium free energy \( F^0 \), show that

\[
F - F^0 = \tau_R \sum_i p_i \log \left( \frac{p_i}{p_i^0} \right)
\]

(Hint: add and subtract \( \sum_i p_i \log (p_i^0) \)).

\[
F = \sum_i p_i \varepsilon_i + \tau_R \sum_i p_i \log p_i^0 = \tau_R \sum_i \frac{\varepsilon_i}{z_R} + \tau_R \sum_i p_i \log p_i^0
\]

Add and subtract \( \tau_R \sum_i p_i \log p_i^0 = + \tau_R \sum_i p_i \left( -\frac{\varepsilon_i}{z_R} + \log \frac{z_R}{z_R} \right) \)

\[
F = \sum_i p_i \varepsilon_i + \tau_R \sum_i p_i \log p_i^0 - \tau_R \sum_i p_i \log z
\]

but \(-\tau_R \log z = F^0\)

\[
\Rightarrow \quad F - F^0 = \tau R \sum_i p_i \log \left( \frac{p_i}{p_i^0} \right)
\]

f) (3 points) Using again the fact that \( \log(x) < (x - 1) \), show that \( F \geq F^0 \), with the equality being only obtained if \( p_i = p_i^0 \) for all \( i \). The free energy is minimum at equilibrium!

\[
F^0 - F = \tau_R \sum_i p_i \log \left( \frac{p_i}{p_i^0} \right) \leq \tau_R \sum_i p_i \frac{p_i^0 - 1}{p_i^0} = 0
\]

\[
\Rightarrow \quad F^0 \leq F
\]

with equality if all the

\[
p_i = p_i^0
\]
5) Black body radiation pressure (25 points)

We consider photons in thermal equilibrium in a cavity at temperature $T$. We would like to compute the radiation pressure that they exert on the wall of the cavity.

a) (5 points) Show that the density of states in energy is

$$D(\varepsilon) \, d\varepsilon = \frac{\varepsilon^2 \, d\varepsilon}{\pi^2 h^3 c^3}$$

we start from

$$\frac{d3x \times d3p}{2} = \frac{8\pi p^2 \, d\varepsilon}{8\pi^3 \hbar^3}$$

$$D(\varepsilon) \, d\varepsilon = \int \frac{2 \, d3p}{\text{angles}} = \frac{8\pi p^2 \, d\varepsilon}{8\pi^3 \hbar^3}$$

$$D(\varepsilon) \, d\varepsilon = \frac{\varepsilon^2 \, d\varepsilon}{\pi^2 h^3 c^3}$$

b) (5 points) With the help of an appropriate sketch, argue that the flux of photons incident at energy $\varepsilon$ at angle $\theta$ within a solid angle $d\Omega$ on a wall surface area $dA$ is

$$\phi d\varepsilon d\Omega dA = \frac{1}{\exp(\frac{\varepsilon}{T}) - 1} \cdot \frac{\varepsilon^2}{4\pi^3 h^3 c^3} \, d\varepsilon \, d\Omega \, dA$$

$$\phi d\varepsilon d\Omega dA = \frac{1}{\exp(\frac{\varepsilon}{T}) - 1} \cdot \frac{\varepsilon^2}{4\pi^3 h^3 c^3} \, d\varepsilon \, d\Omega \, dA$$

$$\Rightarrow \phi \, d\varepsilon \, d\Omega \, dA = \frac{1}{\exp(\frac{\varepsilon}{T}) - 1} \cdot \frac{\varepsilon^2}{4\pi^3 h^3 c^3} \, d\varepsilon \, d\Omega \, dA$$

(c) (5 points) From this result compute the pressure and compare it with the total energy density $u$.

Assume each photon scatters specularly. The change of momentum for each photon is

$$\Delta p = 2 \varepsilon \cos \theta = \frac{2 \varepsilon}{c}$$

and $\phi \, d\varepsilon \, d\Omega \, dA \, d\varepsilon \, dA$ are scattering.

$$\Rightarrow \text{Pressure} = \frac{\text{force}}{dA} = \frac{\Sigma \Delta p}{dt \cdot dA} = \int \frac{2 \varepsilon}{c} \, \frac{1}{\exp(\frac{\varepsilon}{T}) - 1} \cdot \frac{\varepsilon^2}{4\pi^3 h^3 c^3} \, d\varepsilon \, d\Omega \, dA$$

$$P = \frac{1}{3} \int \frac{1}{\exp(\frac{\varepsilon}{T}) - 1} \cdot \frac{\varepsilon^3}{\pi^2 h^3 c^3} \, d\varepsilon = \frac{1}{3} u$$
d) (10 points) Compare with the value that you get by taking the appropriate derivative of the Helmholtz Free Energy. We did that in midterm 2: compute the Free Energy for one mode from the partition function and some over the modes. A simple integration by part give you the result you want.

\[
Z = \sum_{s=0}^{\infty} e^{-\frac{s \varepsilon}{\kappa}} = \frac{1}{1 - e^{-\frac{\varepsilon}{\kappa}}}
\]

\[
F_\varepsilon = -T \log Z_\varepsilon
\]

\[
\Rightarrow F = \int F_\varepsilon \Gamma \delta (\varepsilon) d\varepsilon
\]

\[
F = \int \frac{\partial F}{\partial V} \left( -\int \log \left( \frac{1}{1 - e^{-\frac{\varepsilon}{\kappa}}} \right) \frac{\varepsilon^2}{\pi^2 \kappa^2} d\varepsilon \right)
\]

we integrate by part \( \int u d\varepsilon = [uv] - \int v du \) with \( dv = \varepsilon^2 d\varepsilon \)

\[
F = \left[ -T \log \left( 1 - e^{-\frac{\varepsilon}{\kappa}} \right) \frac{\varepsilon^3}{3 \pi^2 \kappa^2} \right]_0^\infty - \frac{1}{3} \int \frac{-T e^{-\frac{\varepsilon}{\kappa}}}{1 - e^{-\frac{\varepsilon}{\kappa}}} \frac{\varepsilon^3}{\pi^2 \kappa^2} d\varepsilon
\]

\[
= \frac{1}{3} \int \frac{1}{\varepsilon^2} \frac{\varepsilon^3}{\pi^2 \kappa^2} = \frac{1}{3}
\]
6) Elasticity of DNA. (35 points)

Biophysicists are now able to attach a DNA strand at one end to a glass plate and at the other to a small ferromagnetic bead (figure a). With a magnet, they can then apply a force $F$ perpendicular to the glass plate to the DNA and measure the displacement of the bead (that they can see under the microscope) and deduce the elasticity of a single DNA molecule.

![Figure a](image_a)

![Figure b](image_b)

To first order such a DNA molecule can be described as made of $N$ links (the bases) of length $b$, which can orient themselves in random direction with respect to each other. The molecules of the solution tend to curl up the DNA through collisions (figure b) while the force $F$ tends to straighten it.

a) (5 points) Consider one such link oriented at an angle $\theta$ with respect to Ox, the direction of the force. Show that the potential energy of the link (fixing $V=0$ for $\theta=0$) is

$$
\epsilon_p = bF(1 - \cos \theta)
$$

$$
\epsilon_p = \int_0^\theta \sum F \sin \theta \, d\theta = \int F \frac{b}{2} \sin \theta - F \left( -\frac{b}{2} \right) \sin \theta \, d\theta
$$

$$
= \int bF \sin \theta \, d\theta = bF \left[ -\cos \theta \right]_0^\theta = bF (1 - \cos \theta)
$$

b) (10 points) Assuming that the density of quantum states per unit solid angle is constant, show that the partition function for such a link has the form

$$Z_i = \frac{K \tau}{Fb} \sum \frac{Fb}{\tau} \sinh \left( \frac{Fb}{\tau} \right) Z_{kin}
$$

where $K$ is a constant and $Z_{kin}$ the kinetic energy part that we do not need to compute.

$$
Z_1 = k_1 \int d\psi \int d\cos \theta \ e^{-\frac{\epsilon}{\tau}} \ e^{-\frac{Fb \cos \theta}{\tau}} Z_{kin}
$$

$$
= k_1 \tau e^{-\frac{Fb}{\tau}} \int d\cos \theta \ e^{\frac{Fb \cos \theta}{\tau}} Z_{kin}
$$

$$
= 4\pi k_1 \frac{\tau}{Fb} \int e^{-\frac{Fb}{\tau} \sin \theta} h \left( \frac{Fb}{\tau} \right) Z_{kin}
$$

of the form...
c) (15 points) By taking the appropriate derivative of log $Z_i$, show the mean distance $\langle r \rangle$ of the bead from the glass plate when we exert a force $F$ is

$$\langle r \rangle = \frac{-N\tau}{F} + N\coth \left( \frac{bF}{\tau} \right)$$

The displacement for one individual bead is

$$\delta x = \frac{b \cos \Theta}{\tau} - \frac{bF(1 - \cos \Theta)}{\tau^2}$$

Let us look at

$$\frac{\partial \log Z_{ie}}{\partial \tau} = \frac{F}{Z_{ie}} \left( b \cos \Theta - \langle x \rangle \right)$$

$$\Rightarrow \langle x \rangle = \frac{b - \frac{b^2}{2}}{F} \frac{\partial \log Z_{ie}}{\partial \tau} = \frac{F}{Z_{ie}} \left( \frac{1}{\tau} + \frac{Fb}{\tau^2} - \frac{Fb \cosh Fb}{Z_{ie}} \right)$$

$$\Rightarrow \langle x \rangle = v_{ss}$$

d) (5 points) Explore the limits when the force $F$ is very large and very small. Identify the quantities that play the role of pressure and volume for this one-dimensional system. What is in both cases the equivalent of the equation of state for an ideal gas

$$PV = N\tau$$

Note: for small $x$, $\coth(x) \approx \frac{1}{x} \left( 1 + \frac{x^2}{3} + \ldots \right)$

$F$ plays the role of the pressure.

$\langle x \rangle$ or $b - \langle x \rangle$ plays the role of the volume.

a) If $F$ is very large, $\coth \left( \frac{bF}{\tau} \right) \rightarrow 1$, $\langle x \rangle \rightarrow 1$.

$$\langle x \rangle = \frac{N\tau}{F} - \frac{N\tau}{F}$$

$$F \left( b - \langle x \rangle \right) = \frac{N\tau}{F}$$

Similar to $PV = N\tau$.

b) If $F$ is very small, $\langle x \rangle \rightarrow \frac{N\tau}{F}$.

$$\langle x \rangle = \frac{N\tau^2 F}{3}$$

very different qualitative tendency to bundle.
7) **Shottky barrier between two metals (35 points).**

We consider two metals of different Fermi level $\varepsilon_{F1}$ and $\varepsilon_{F2}>\varepsilon_{F1}$. We will describe the electrons in these metals as ideal quantum gases at zero temperature.

a) (5 points) We bring these two metals in contact. Describe qualitatively the situation in equilibrium. Describe in particular the balance between drift and diffusion currents at equilibrium (i.e. what stops the flow of electrons from one metal to the other). Using this argument, plot below the net charge density (number density of fixed positive ions - number density of mobile electrons) as a function of the position $x$ on an axis perpendicular to the interface (the interface is at $x=0$).

![Diagram of Shottky barrier](image)

The electrons of metal 2 which are close to the interface will tend to go to metal 1, leaving behind positive charge and creating an excess of negative charge in metal 1. As a result 2 becomes more positive and 1 becomes more negative. This causes an electric field (in the direction opposite to the Ox axis) close to the interface. This field will eventually balance the electron diffusion from side 2 to side 1.

1) Keep electrons in metal 1 close to the interface.
2) Prevent electrons in metal 2 to come close to the interface.
3) Prevent electric field from neutralizing the positive charge.

At a distance far from the interface, there is no electric field so the positive and negative charges shield each other.
b) (5 points) Deduce from this the qualitative shape of the electric potential shown below. Plot the electron maximum energy at equilibrium as a function of $x$.

Since $V(x) = -\int E \cdot dx$, it will have a 5-shape as shown on the left plot.

The Fermi surface $\varepsilon_F (x) = \varepsilon_F + q_e V(x) = \varepsilon_F - |q_e| V_{tot}$ has therefore the shape shown at the right.

The electron maximum energy is $\varepsilon_{tot}$ and is therefore $V_2 - V_1 = \frac{(\varepsilon_{F_2} - \varepsilon_{F_1})}{-q_e}$

(c) (5 points) We now describe this situation quantitatively. Starting from what you know about chemical potentials, show that the difference of voltage between the two metals far from the interface is

$$V_2 - V_1 = \frac{(\varepsilon_{F_2} - \varepsilon_{F_1})}{-q_e}$$

$q_e$ is the charge of the electron i.e. $q_e > 0$.

The total chemical potential is constant.

At $-\infty$ metal 1 is not disturbed and its internal chemical potential is therefore $\varepsilon_{F_1}$

$$\mu_{tot} = \varepsilon_{F_2} + q_e V_1$$

Similarly at $+\infty$

$$\mu_{tot} = \varepsilon_{F_2} + q_e V_2$$

$$\Rightarrow V_2 - V_1 = \frac{(\varepsilon_{F_2} - \varepsilon_{F_1})}{-q_e}$$

(d) (5 points) What is the relation between the internal chemical potential $\mu_{int}(x)$ (local Fermi level) and the density of electrons $n(x)$? For simplicity we assume the same effective mass $m$ of the electrons in the two media.

We have a Fermi-Dirac at $\varepsilon = 0$

$$\int_0^{\mu_{int}} \theta(\varepsilon) \, d\varepsilon = n(x)$$

$$n = \int_0^{\mu_{int}} \theta(\varepsilon) \, d\varepsilon$$

$$\int_0^{\mu_{int}} \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon} \, d\varepsilon$$

$$\mu_{int} = \frac{h^2}{2m} \left( \frac{3\pi^2 n(x)}{2} \right)^{2/3}$$

$$n(x) = \frac{1}{\frac{3\pi^2}{2}} \left( \frac{2m \mu_{int}}{h^2} \right)^{3/2}$$
e) (5 points) How is the electric potential $V(x)$ related to the density of positive and negative charges? We call the total dielectric constant $\varepsilon$.

\[
\frac{\partial^2 V}{\partial x^2} = \frac{q}{\varepsilon} = -\left(\frac{q^+_n(x) + q^+_n(x)}{\varepsilon}\right)
\]

but \( q^+_n = -q^_e \)

\[
\frac{\partial^2 V}{\partial x^2} = q^e \left( \frac{n^+_n(x) - n^e_n(x)}{\varepsilon} \right)
\]

f) (5 points) How is the local density of positive charges related to the (unperturbed) Fermi levels $\varepsilon_{F1}$ and $\varepsilon_{F2}$?

The density of the positive charges is the same as the density of negative charges in the unperturbed model.

\[
\Rightarrow \quad n^+_n(x) = \frac{1}{3\pi^2} \left( \frac{2m_{e1} \varepsilon_{F1}}{h^2} \right)^{3/2} \quad n^+_n(x) = \frac{1}{3\pi^2} \left( \frac{2m_{e2} \varepsilon_{F2}}{h^2} \right)^{3/2}
\]

g) (5 points) Bringing all of this together

\[
\frac{\partial^2 [V(x) - V_1]}{\partial x^2} = \frac{q}{\varepsilon} \frac{1}{3\pi^2} \left( \frac{2m_{e1} \varepsilon_{F1}}{h^2} \right)^{3/2} \left\{ \left[ 1 - \frac{q}{\varepsilon_{F1}} \right] V(x) - V_1 \right\} \quad x < 0
\]

\[
\frac{\partial^2 [V(x) - V_2]}{\partial x^2} = \frac{q}{\varepsilon} \frac{1}{3\pi^2} \left( \frac{2m_{e2} \varepsilon_{F2}}{h^2} \right)^{3/2} \left\{ \left[ 1 - \frac{q}{\varepsilon_{F2}} \right] V(x) - V_2 \right\} \quad x > 0
\]

which indeed go to zero at minus and plus infinity.

Let us write

\[
\mu_{F1} = \varepsilon_{F1} + q^e V_1
\]

\[
\mu_{F2} = \varepsilon_{F1} - q^e (V(x) - V_1)
\]

\[
f_{a x < 0} \quad \frac{\partial^2 V(x)}{\partial x^2} = \frac{q^e}{\varepsilon} \frac{1}{3\pi^2} \left( \frac{2m_{e1} \varepsilon_{F1}}{h^2} \right)^{3/2} \left\{ \left[ 1 - \frac{q}{\varepsilon_{F1}} \right] V(x) - V_1 \right\}
\]

By factoring out

\[
\frac{1}{3\pi^2} \left( \frac{2m_{e1} \varepsilon_{F1}}{h^2} \right)^{3/2}
\]

we have as written above

\[
f_{a x < 0} \quad \frac{\partial^2 (V(x) - V_1)}{\partial x^2} = \frac{q^e}{\varepsilon} \frac{1}{3\pi^2} \left( \frac{2m_{e1} \varepsilon_{F1}}{h^2} \right)^{3/2} \left\{ 1 - \left[ 1 - \frac{q}{\varepsilon_{F1}} \right] (V(x) - V_1) \right\}
\]

and a similar result for $x > 0$. 

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h) Additional credit (10 points) These equations do not have a close form solution. However, for small relative differences of Fermi levels, they can be linearized:

\[
\frac{\partial^2 [V(x) - V_i]}{\partial x^2} = \frac{q^2}{\varepsilon} \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon_{F1}} [V(x) - V_i] \quad x < 0
\]

\[
\frac{\partial^2 [V(x) - V_i]}{\partial x^2} = \frac{q^2}{\varepsilon} \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon_{F2}} [V(x) - V_i] \quad x > 0
\]

By taking into account the continuity of the potential and field at the interface show that

\[
V(x) = V_i + (V_2 - V_i) \frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}} \exp\left(\sqrt{a}x\right) \quad \text{for } x < 0
\]

\[
V(x) = V_2 - (V_2 - V_i) \frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}} \exp\left(-\sqrt{b}x\right) \quad \text{for } x > 0
\]

with

\[
a = \frac{q^2}{\varepsilon} \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon_{F1}} \quad b = \frac{q^2}{\varepsilon} \frac{1}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\varepsilon_{F2}}
\]

We just use \((1 + x)^{3/2} \div 1 + \frac{3}{2}x\) and get

the equations above, of the form

\[
\frac{a^2 (V(x) - V_i)}{\sqrt{a}x} = a \left[ V(x) - V_i \right] \quad x < 0
\]

\[
\frac{b^2 (V(x) - V_2)}{\sqrt{b}x} = b \left[ V(x) - V_2 \right] \quad x > 0
\]

The general solution is

\[
V(x) - V_i = A_1 e^{-\sqrt{a}x} + B_1 e^{\sqrt{a}x}
\]

Clearly \(A_1 = 0\) to have

\[
V(x) \to V_i \quad \text{when} \quad x \to 0
\]

\[
V(x) - V_2 = A_2 e^{-\sqrt{b}x} + B_2 e^{\sqrt{b}x}
\]

And we need \(B_2 = 0\) for \(V(x) \to V_2 \quad \text{for } x \to 0\)

We used them to impose the continuity of the potential and its derivative at the interface

\[
V(0) = V_1 + B_1 = V_2 + A_2
\]

\[
\frac{dV}{dx}(0) = B_1 \sqrt{a} = -A_2 \sqrt{b}
\]

\[
\Rightarrow \quad B_1 = \frac{(V_2 - V_1) \sqrt{b}}{\sqrt{a} + \sqrt{b}} \quad A_2 = \frac{(V_2 - V_1) \sqrt{a}}{\sqrt{a} + \sqrt{b}}
\]

\[
\frac{1}{\sqrt{a}} \quad \text{is of the order of Angstroms.}
\]

Very narrow barriers through which the carriers can tunnel easily.