The Black Body Radiation

= Chapter 4 of Kittel and Kroemer

The Planck distribution
Derivation

Black Body Radiation
Cosmic Microwave Background
The genius of Max Planck

Other derivations

Stefan Boltzmann law
Flux => Stefan-Boltzmann
Example of application: star diameter

Detailed Balance: Kirchhoff laws

Another example: Phonons in a solid
Examples of applications
Study of Cosmic Microwave Background
Search for Dark Matter
The Planck Distribution

Photons in a cavity

Mode characterized by

<table>
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<tr>
<th>Frequency $v$</th>
<th>Angular frequency $\omega = 2\pi v$</th>
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Number of photons $s$ in a mode $\Rightarrow$ energy $\varepsilon = s\hbar \nu = s\hbar \omega$

Occupation number

$$\langle s \rangle = \frac{1}{\exp \left( \frac{\hbar \omega}{\tau} \right) - 1}$$

Radiation energy density between $\omega$ and $\omega + d\omega$?

2 different polarizations

$$u_\omega d\omega = \frac{\hbar \omega^3 d\omega}{\pi^2 c^3 \left( \exp \left( \frac{\hbar \omega}{\tau} \right) - 1 \right)} = u_\nu dv = \frac{8\pi \nu^3 dv}{c^3 \left( \exp \left( \frac{\hbar \nu}{\tau} \right) - 1 \right)}$$
Cosmic Microwave Radiation

Big Bang => very high temperatures!
When $T \approx 3000K$, $p+e$ recombine $\Rightarrow$ H and universe becomes transparent
Redshifted by expansion of the universe: 2.73K
Other derivations

Grand canonical method

\[ \langle s \rangle = \langle N \rangle = \frac{\partial (\tau \log Z)}{\partial \mu} = \frac{1}{\exp\left(\frac{\hbar \omega - \mu}{\tau}\right) - 1} \]

What is \( \mu \)?

\( \gamma + e \leftrightarrow e \quad \mu \gamma + \mu_e - \mu_e = 0 \Rightarrow \mu \gamma = 0 \)

Microcanonical method

To compute mean number photons in one mode, consider ensemble of \( N \) oscillators at same temperature and compute total energy \( U \).

We have to compute the multiplicity \( g(n,N) \) of number of states with energy \( U \), number of combinations of \( N \) positive integers such that their sum is \( n \)

\[ \frac{1}{\tau} = \frac{\partial \sigma}{\partial U} = \frac{1}{\hbar \omega} \log \left( 1 + \frac{N \hbar \omega}{U} \right) = \frac{1}{\hbar \omega} \log \left( 1 + \frac{1}{\langle s \rangle} \right) \]

\[ \langle s \rangle = \frac{1}{\exp\left(\frac{\hbar \omega}{\tau}\right) - 1} \]
**Fluxes**

**Energy density traveling in a certain direction**

So far energy density integrated over solid angle. If we are interested in energy density traveling traveling in a certain direction, isotropy implies

\[ u_\omega(\theta, \varphi) d\omega d\Omega = \frac{\hbar \omega^3 d\omega d\Omega}{4\pi^2 c^3 \left( \exp\left( \frac{\hbar \omega}{\tau} \right) - 1 \right)} \]

Note if we use \( \nu \) instead of \( \omega \)

\[ u_\nu(\theta, \varphi) d\nu d\Omega = \frac{2h \nu^3 d\nu d\Omega}{c^3 \left( \exp\left( \frac{h \nu}{\tau} \right) - 1 \right)} \]

**Flux density in a certain direction=brightness**

(Energy /unit time, area, solid angle,frequency) opening is perpendicular to direction

\[ I_\nu d\nu dA d\Omega = \frac{1}{dt} d\nu u_\nu c dtdA d\Omega = \frac{2h \nu^3 d\nu dA d\Omega}{c^2 \left( \exp\left( \frac{h \nu}{\tau} \right) - 1 \right)} \]

**Flux density through a fixed opening**

(Energy /unit time, area,frequency)

\[ J_\nu d\nu = \frac{1}{dt} d\nu \int_0^{2\pi} d\varphi \int_0^1 u_\nu c d\vartheta dA \theta d\vartheta = \frac{2\pi h \nu^3 d\nu dA}{c^2 \left( \exp\left( \frac{h \nu}{\tau} \right) - 1 \right)} = \frac{c}{4} u_\nu d\nu dA \]

\( \nu \) energy in cylinder

\( d\nu \) energy in cylinder

\( c\theta \)
Stefan-Boltzmann Law

Total Energy Density

Integrate on $\omega$

\[ u = \frac{\pi^2}{15\hbar^3 c^3} \tau^4 = a_B T^4 \]

with \( a_B = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} \)

Total flux through a fixed aperture***

multiply above result by $c/4$

\[ J = \frac{\pi^2}{60\hbar^3 c^2} \tau^4 = \sigma_B T^4 \]

with \( \sigma_B = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} = 5.67 \times 10^{-8} \text{ W/m}^2/\text{K}^4 \)

Stefan-Boltzmann constant!
**Definition:** A body is black if it absorbs all electromagnetic radiation incident on it. Usually true only in a range of frequency.

- e.g. A cavity with a small hole appears black to the outside.

**Detailed balance:** in thermal equilibrium, power emitted by a system = power received by this system! Otherwise temperature would change!

**Consequence:** The spectrum of radiation emitted by a black body is the “black body” spectrum calculated before: $u_{BB}(\omega)d\omega = u_{cavity}(\omega)d\omega$

**Absorptivity, Emissivity:**
- Absorptivity = fraction of radiation absorbed by body
- Emissivity = ratio of emitted spectral density to black body spectral density.

**Kirchhoff: Emissivity = Absorptivity** $a(\omega) = e(\omega)$
Entropy, Number of photons

**Entropy**

\[ \sigma = \int \sigma_{\omega} d^3x \frac{\omega^2 d\omega}{\pi^2 c^3} \]

\[ \sigma_{\omega} = -\sum_s p_s \log p_s \]

**Number of photons**

\[ N_\gamma = \frac{V 2\zeta(3)}{\pi^2 c^3 \hbar^3} \tau^3 = \frac{30\zeta(3)}{\pi^4} \frac{a_B}{k_B} VT^3 \approx 0.37 \frac{a_B}{k_B} VT^3 \]

**Proportional**

\[ \sigma \approx 3.6 N_\gamma \]
Rayleigh-Jeans region (= low frequency)

If $\hbar \omega \ll \tau$  \( \langle \varepsilon_\omega \rangle \approx \tau \)

Power / unit area/solid angle/unit frequency

= Brightness

\( I_v d\nu d\Omega d\omega \approx \tau \frac{\omega^2 d\omega}{\pi^2 c^3} \)

Power emitted / unit (fixed) area

\( J_\omega d\omega = J_v d\nu \approx \tau \frac{\omega^2 d\omega}{4\pi^2 c^2} = \tau \frac{2\pi d\nu}{\lambda^2} \)

\( \frac{dP}{d\nu} = I_v \Omega_e A_e = \frac{2\pi \Omega_e A_e}{\lambda^2} = \frac{2\pi \Omega_r A_r}{\lambda^2} = 2\tau \)

Detected Power (1 polarization)

\( \frac{dP}{d\nu} \) (1 polarization) = \( \tau \)

Antenna temperature

\( T_A = \frac{1}{k_B} \frac{dP}{d\nu} \) (1 polarization)

\( A_e = \Omega_r d^2 \)

\( \Omega_e = \frac{A_r}{d^2} \)

If diffraction limited:

\( \Omega_r A_r = \lambda^2 \)
## Applications

### Many!

**e.g. Star angular diameter**

Approximately black body and spherical!
- Spectroscopy $\Rightarrow$ Effective temperature
- Apparent luminosity $l = $ power received per unit area

$$l = \frac{L \Omega_e}{4\pi} \frac{1}{A_r} = \frac{L}{4\pi d^2}$$

But power output

$$L = 4\pi r^2 \sigma B T_{\text{eff}}^4$$

$$\Rightarrow l = \frac{r^2}{d^2} \sigma B T_{\text{eff}}^4$$

Angular diameter

$$\text{angular diameter} = \frac{r_\perp}{d} = \sqrt{\frac{l}{\sigma B T_{\text{eff}}^4}}$$

$\Rightarrow$ Baade-Wesserlink distance measurements of varying stars
- Oscillating stars (Cepheids, RR Lyrae)
- Supernova
  - assuming spherical expansion
Phonons in a solid

Phonons:
Quantized vibration of a crystal described in same way as photons
If s phonons in a mode
Same as for photons but 3 modes
Maximum energy (minimum wavelength/finite # degrees of freedom)

\[ \varepsilon = \hbar \omega \quad p = \hbar k = \frac{\hbar \omega}{c_s} \quad \varepsilon_s = s\hbar \omega \]

\[ \langle s \rangle = \frac{1}{\exp\left(\frac{\hbar \omega}{\tau}\right) - 1} \]

Debye approximation:

- isotropic
- \[ k = \frac{\omega}{c_s} \quad \text{or} \quad p = \frac{\varepsilon}{c_s} \]

Introducing the Debye temperature \( T_D = \theta = \frac{\hbar c_s}{k_B} \left( 6\pi^2 \frac{N}{V} \right)^{1/3} = \frac{\hbar \omega_D}{k_B} \)

\[ U = \frac{3}{5} \pi^4 k_B N \frac{T^4}{T_D^3} \]
\[ C_V = \frac{12}{5} \pi^4 k_B N \left( \frac{T}{T_D} \right)^3 \]
\[ \sigma = \frac{12}{15} \pi^4 k_B N \left( \frac{T}{T_D} \right)^3 \]
Applications

**Calorimetry:** Measure energy deposition by temperature rise

\[ \Delta T = \frac{\Delta E}{C} \Rightarrow \text{need small } C \]

**Bolometry:** Measure energy flux \( F \) by temperature rise

Chopping e.g., between sky and calibration load

\[ \Delta T = \frac{\Delta F}{G} \Rightarrow \text{need small } G \text{ but time constant } \frac{C}{G} \text{ limited by stability } \Rightarrow \text{small heat capacity } C \]

Very sensitive!

Heat capacity goes to zero at low temperature \( C \approx T^3 \)

Study of cosmic microwave background
Search for dark matter particles